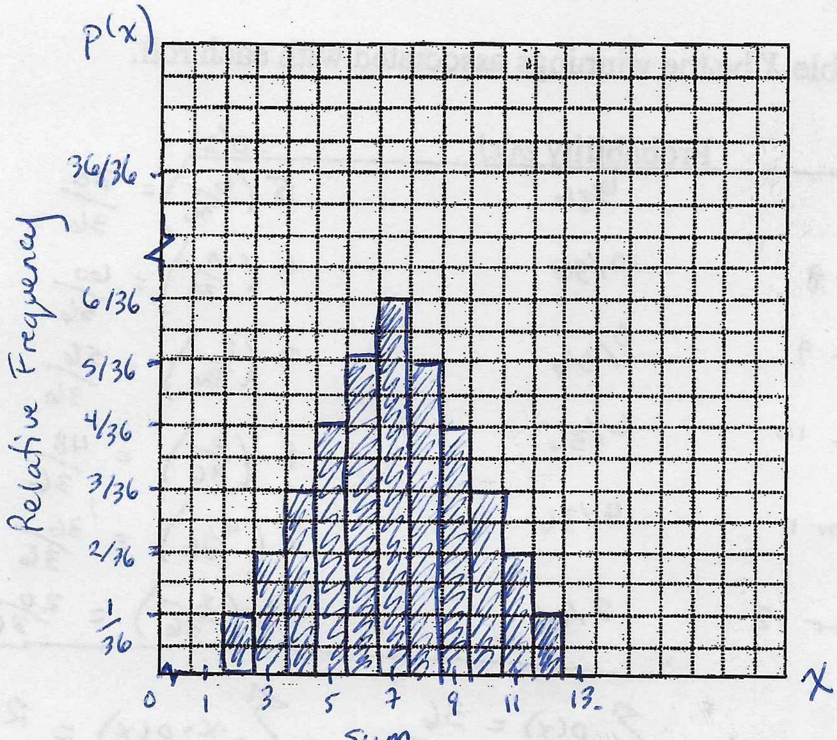
Expected Value ; Probability Distributions

The following represents the graphical model (i.e., a histogram) with probability (i.e., relative frequency) plotted against sum.



Expected Value of a Discrete Random Variable

Once a probability distribution has been defined, then this model can be used to further analyze an experiment. A useful piece of information that can be obtained from the distribution is the **expected value or expectation.** This value represents the quantity that you can expect to obtain when an experiment is performed and is denoted by **E(x).** Let's examine this concept with the following example.

E.g., 2. A game is defined by the rules that two dice are rolled and the player wins varying amounts depending on the sum of the two dice according to the table below.

Sum	Winnings ($)	Sum	Winnings
2	10	7	5
3	9	8	6
4	8	9	7
5	7	10	8
6	6	11	9
		12	10

a) What can a player expect to win by playing this game? $E(x) = \sum x \cdot p(x)$
b) What would be a fair value to pay to play this game?

Solution:
a) Let the random variable *X* be the winnings associated with each roll.

Winnings (x)	Sum	Probability p(x)	xp(x)
5	7	6/36	$5\left(\frac{6}{36}\right) = \frac{30}{36}$
6	6 or 8	10/36	$6\left(\frac{10}{36}\right) = \frac{60}{36}$
7	5 or 9	8/36	$7\left(\frac{8}{36}\right) = \frac{56}{36}$
8	4 or 10	6/36	$8\left(\frac{6}{36}\right) = \frac{48}{36}$
9	3 or 11	4/36	$9\left(\frac{4}{36}\right) = \frac{36}{36}$
10	2 or 12	2/36	$10\left(\frac{2}{36}\right) = \frac{20}{36}$
		$\sum p(x) = \frac{36}{36} = 1$	$\sum x \cdot p(x) = \frac{250}{36}$

a) A player could expect to win \$6.94.

b) If you were to play this game, you would likely not want to pay more than what you could win. Thus, a fair value to pay should be no more than the expected value — i.e., \$6.49.

In the previous example, we calculated the expected value as given by

$$E(x) = \sum_x x \cdot p(x)$$

Below, you will find the expansion for the above function.

Expected Value of a Discrete Random Variable

The expected value of a discrete random variable, X, is the sum of the terms of the form $X \cdot P(X)$ for all possible values of X. In other words, if X takes on the values $x_1, x_2, \ldots, x_n$, then the expected value of X is given by

$$E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + \ldots + x_n P(X = x_n)$$
$$= \sum_{i=1}^{n} x_i P(X = x_i)$$

where n represents the number of terms in the sum.

Now, we will further explore the concept of expected value with a few examples.

E.g., 3. Suppose you were to toss three coins.

a) What is the likelihood (probability) that you would observe at least two heads?

Let X represent the number of heads observed)

$$P(X \geq 2) = P(2H, 1T) + P(3H)$$
$$= \frac{3}{8} + \frac{1}{8}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$

$\therefore$ There is a $\frac{1}{2}$ likelihood of observing at least 2 heads.

b) What is the expected number of heads?

To determine the expected number of heads, you need to incorporate the probability distribution for the number of heads. ∴

Let X be the number of heads.

\# of heads

Prob. of this # of heads

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3)$$

$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8}$$

$$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} \text{ or } 1.5$$

∴ The expected number of heads is 1.5.

E.g., 4. A committee of four people is to be chosen randomly from four males and six females. What is the expected number of females on the committee?

Solution: To find the expectation, first determine the probability distribution. Let the random variable *X* be the number of females on the committee. In this situation, we know that $X=\{0, 1, 2, 3, 4\}$.

\# of committees $\overline{w}$ no restrictions $= \binom{10}{4} = 210$

$n(X=0) = \binom{4}{4}\binom{6}{0} = 1$

$n(X=1) = \binom{4}{3}\binom{6}{1} = 24$

$n(X=2) = \binom{4}{2}\binom{6}{2} = 90$

$n(X=3) = \binom{4}{1}\binom{6}{3} = 80$

$n(X=4) = \binom{4}{0}\binom{6}{4} = 15$

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3) + 4 \cdot P(X=4)$$

$$= 0 \cdot \frac{1}{210} + 1 \cdot \frac{24}{210} + 2 \cdot \frac{90}{210} + 3 \cdot \frac{80}{210} + 4 \cdot \frac{15}{210}$$

$$= 0 + \frac{24}{210} + \frac{180}{210} + \frac{240}{210} + \frac{60}{210} = \frac{504}{210} \doteq 2.4$$

∴ One would expect to see at least 2 females on the committee.